

On \mathcal{D} -equivalence classes of some graphs

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Abstract

Let G be a simple graph of order n . The domination polynomial of G is the polynomial $D(G, x) = \sum_{i=1}^n d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of G of size i . The n -barbell graph Bar_n with $2n$ vertices, is formed by joining two copies of a complete graph K_n by a single edge. We prove that for every $n \geq 2$, Bar_n is not \mathcal{D} -unique, that is, there is another non-isomorphic graph with the same domination polynomial. More precisely, we show that for every n , the \mathcal{D} -equivalence class of barbell graph, $[Bar_n]$, contains many graphs, which one of them is the complement of book graph of order $n-1$, B_{n-1}^c . Also we present many families of graphs in \mathcal{D} -equivalence class of $K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_k}$.

Keywords: Domination polynomial; \mathcal{D} -unique; Equivalence; Generalize barbell graphs.

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1 Introduction

All graphs in this paper are simple of finite orders, i.e., graphs are undirected with no loops or parallel edges and with finite number of vertices. The *complement* G^c of a graph G , is a graph with the same vertex set as G and with the property that two vertices are adjacent in G^c if and only if they are not adjacent in G . For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V(G)$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S . The *domination number* $\gamma(G)$, is the minimum cardinality of a dominating set in G . For a detailed treatment of domination theory, the reader is referred to [10].

Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. The *domination polynomial* $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$ (see [2, 6, 11]). This polynomial is the generating polynomial for the number of dominating sets of each cardinality.

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Calculating the domination polynomial of a graph G is difficult in general, as the smallest power of a non-zero term is the domination number $\gamma(G)$ of the graph, and determining whether $\gamma(G) \leq k$ is known to be NP-complete [9]. But for certain classes of graphs, we can find a closed form expression for the domination polynomial. Two graphs G and H are said to be *dominating equivalent*, or simply \mathcal{D} -equivalent, written $G \sim H$, if $D(G, x) = D(H, x)$. It is evident that the relation \sim of being \mathcal{D} -equivalence is an equivalence relation on the family \mathcal{G} of graphs, and thus \mathcal{G} is partitioned into equivalence classes, called the *\mathcal{D} -equivalence classes*. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} : H \sim G\}.$$

We call $[G]$ the equivalence class determined by G . A graph G is said to be dominating unique, or simply \mathcal{D} -unique, if $[G] = \{G\}$ [3]. Determining \mathcal{D} -equivalence class of graphs is one of the interesting problems on equivalence classes.

A question of recent interest concerning this equivalence relation $[\cdot]$ asks which graphs are determined by their domination polynomial. It is known that cycles [2] and cubic graphs of order 10 [7] (particularly, the Petersen graph) are, while if $n \equiv 0 \pmod{3}$, the paths of order n are not [2]. In [8], authors completely described the complete r -partite graphs which are \mathcal{D} -unique. Their results in the bipartite case, settles in the affirmative a conjecture in [1].

Let n be any positive integer and Bar_n be n -barbell graph with $2n$ vertices which is formed by joining two copies of a complete graph K_n by a single edge. In this paper, we consider n -barbell graphs and study their domination polynomials. We prove that for every $n \geq 2$, Bar_n is not \mathcal{D} -unique. More precisely, in Section 2, we show that for every n , $[Bar_n]$ contains many graphs, which one of them is $2K_n$ and another one is the complement of book graph of order $n - 1$, B_{n-1}^c . In Section 3, we present many graphs in $[K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_k}]$.

2 \mathcal{D} -equivalence classes of some graphs

In this section, we study the \mathcal{D} -equivalence classes of some graphs. First we consider the domination polynomial of the complement of book graph.

The n -book graph B_n can be constructed by bonding n copies of the cycle graph C_4 along a common edge $\{u, v\}$, see Figure 1.

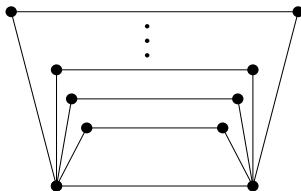


Figure 1: The book graph B_n .

The following theorem gives a formula for the domination polynomial of B_n .

Theorem 2.1 [5] *For every $n \in \mathbb{N}$,*

$$D(B_n, x) = (x^2 + 2x)^n(2x + 1) + x^2(x + 1)^{2n} - 2x^n.$$

Domination polynomials, exploring the nature and location of roots of domination polynomials of book graphs has studied in [5]. Here, we consider the domination polynomial of the complement of the book graphs. We shall prove that the n -barbell graph Bar_n and B_{n-1}^c have the same domination polynomial.

The Turán graph $T(n, r)$ is a complete multipartite graph formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible, and connecting two vertices by an edge whenever they belong to different subsets. The graph will have $(n \bmod r)$ subsets of size $\lceil \frac{n}{r} \rceil$, and $r - (n \bmod r)$ subsets of size $\lfloor \frac{n}{r} \rfloor$. That is, a complete r -partite graph

$$K_{\lceil \frac{n}{r} \rceil, \lceil \frac{n}{r} \rceil, \dots, \lceil \frac{n}{r} \rceil, \lfloor \frac{n}{r} \rfloor, \lfloor \frac{n}{r} \rfloor}.$$

The Turán graph $T(2n, n)$ can be formed by removing a perfect matching, n edges no two of which are adjacent, from a complete graph K_{2n} . As Roberts (1969) showed, this graph has boxicity exactly n ; it is sometimes known as the Robert's graph [13]. If n couples go to a party, and each person shakes hands with every person except his or her partner, then this graph describes the set of handshakes that take place; for this reason it is also called the cocktail party graph. So, the cocktail party graph $CP(t)$ of order $2t$ is the graph with vertices b_1, b_2, \dots, b_{2t} in which each pair of distinct vertices form an edge with the exception of the pairs $\{b_1, b_2\}, \{b_3, b_4\}, \dots, \{b_{2t-1}, b_{2t}\}$. The following result is easy to obtain.

Lemma 2.2 *For every $n \in \mathbb{N}$, $D(CP(n), x) = (1 + x)^{2n} - 2nx - 1$.*

Figure 2 shows the complement of the book graph B_n^c .

The vertex contraction G/u of a graph G by a vertex u is the operation under which all vertices in $N(u)$ are joined to each other and then u is deleted (see[14]).

The following theorem is useful for finding the recurrence relations for the domination polynomials of arbitrary graphs.

Theorem 2.3 [4, 12] *Let G be a graph. For any vertex u in G we have*

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) - (1 + x)p_u(G, x),$$

where $p_u(G, x)$ is the polynomial counting the dominating sets of $G - u$ which do not contain any vertex of $N(u)$ in G .

The following theorem gives a formula for the domination polynomial of the complement of the book graph.

Theorem 2.4 *For every $n \in \mathbb{N}$,*

$$D(B_n^c, x) = ((1 + x)^{n+1} - 1)^2.$$

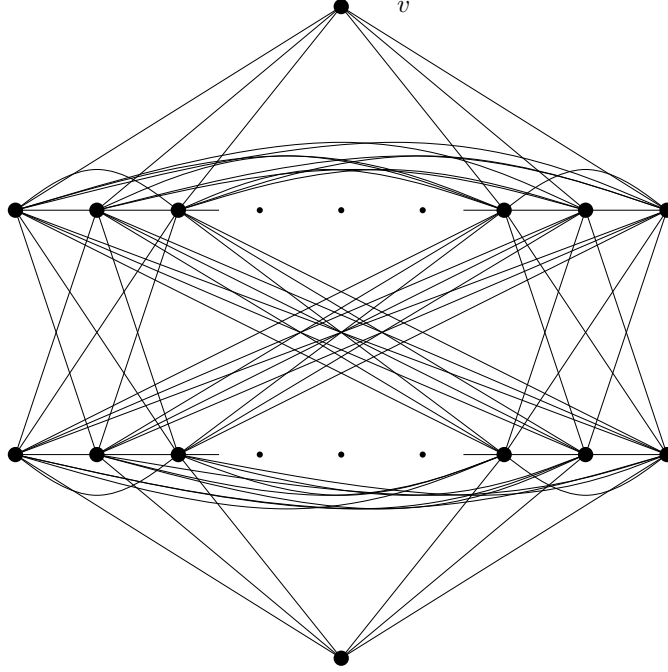


Figure 2: Complement of the book graph B_n^c .

Proof. Consider graph B_n^c and vertex v in the Figure 2. By Theorem 2.3, we have:

$$\begin{aligned}
 D(B_n^c, x) &= xD(B_n^c/v, x) + D(B_n^c - v, x) + xD(B_n^c - N[v], x) - (1+x)p_v(B_n^c, x) \\
 &= (x+1)D(B_n^c - v, x) + xD(K_{n+1}, x) - (1+x)(D(K_{n+1}, x) - (n+1)x - nx^2) \\
 &= (x+1)D(B_n^c - v, x) - D(K_{n+1}, x) + x(1+x)(1+n(1+x)) \\
 &= (x+1)D(B_n^c - v, x) - ((1+x)^{n+1} - 1) + x(1+x)(1+n(1+x)),
 \end{aligned}$$

where $(B_n^c/v) \simeq B_n^c - v$.

Now, we use Theorem 2.3 to obtain the domination polynomial of the graph $B_n^c - v$. We have

$$\begin{aligned}
 D(B_n^c - v, x) &= xD(B_n^c - v/u, x) + D((B_n^c - v) - u, x) \\
 &\quad + xD((B_n^c - v) - N[u], x) - (1+x)p_u(B_n^c - v, x).
 \end{aligned}$$

Since $(B_n^c - v/u) \simeq (B_n^c - v) - u \simeq CP(n)$ and using Lemma 2.2, we have

$$\begin{aligned}
 D(B_n^c - v, x) &= (x+1)D(CP(n), x) + x(D(K_n, x)) - (1+x)(D(K_n, x) - nx) \\
 &= (x+1)D(CP(n), x) - D(K_n, x) + nx(1+x) \\
 &= (x+1)((1+x)^{2n} - (1+2nx)) - ((1+x)^n - 1) + nx(1+x) \\
 &= (1+x)^n((1+x)^{n+1} - 1) - nx(1+x) - x.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
D(B_n^c, x) &= (x+1)((1+x)^n((1+x)^{n+1} - 1) - nx(1+x) - x) \\
&\quad - ((1+x)^{n+1} - 1) + x(1+x)(1+n(1+x)) \\
&= ((1+x)^{n+1} - 1)^2.
\end{aligned}$$

□

The n -barbell graph is the graph on $2n$ vertices which is formed by joining two copies of a complete graph K_n by a single edge, known as a bridge, shown in Figure 3. We denote this graph by Bar_n . For this graph, we shall calculate this domination polynomial. We need the following definition and theorems.

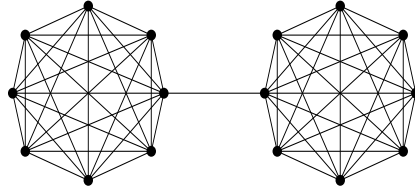


Figure 3: The barbell graph of order 16, Bar_8 .

An irrelevant edge is an edge $e \in E(G)$, such that $D(G, x) = D(G - e, x)$, and a vertex $v \in V(G)$ is domination-covered, if every dominating set of $G - v$ includes at least one vertex adjacent to v in G [12]. We need the following theorems to obtain the domination polynomial of barbell graph Bar_n .

Theorem 2.5 [12] *Let $G = (V, E)$ be a graph. A vertex $v \in V$ of G is domination-covered if and only if there is a vertex $u \in N[v]$ such that $N[u] \subseteq N[v]$.*

Theorem 2.6 [12] *Let $G = (V, E)$ be a graph. An edge $e = \{u, v\} \in E$ is an irrelevant edge in G , if and only if u and v are domination-covered in $G - e$.*

Theorem 2.7 *For every $n \geq 2$ and $n \in \mathbb{N}$,*

$$D(Bar_n, x) = ((1+x)^n - 1)^2.$$

Proof. Let e be an edge joining two K_n in barbell graph. By Theorem 2.5 two end vertices of edge e are domination-covered in $Bar_n - e$. So, by Theorem 2.6 the edge e is an irrelevant edge of Bar_n . Therefore

$$D(Bar_n, x) = D(Bar_n - e, x) = D(K_n \cup K_n, x) = ((1+x)^n - 1)^2.$$

□

The following corollary is an immediate consequence of Theorems 2.4 and 2.7.

Corollary 2.8 *For each natural number n , Bar_n and B_{n-1}^c have the same domination polynomial. More precisely, for every n , $[Bar_n] \supseteq \{Bar_n, B_{n-1}^c, K_n \cup K_n\}$, and $[B_{n-1}^c] \supseteq \{Bar_n, B_{n-1}^c, K_n \cup K_n\}$.*

Here, we present some other families of graphs whose are in the $[Bar_n]$. Let to define the generalized barbell graphs. As we know, the Bar_n is formed by joining two copies of a complete graph K_n by a single edge. We like to join two copies with more edges as follows:

Definition 2.9 *Suppose that $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ are the vertices of two copies of complete graph of order n , K_n and K_n . The generalized barbell graph is denoted by $Bar_{n,t}$ and is a graph with $V(Bar_{n,t}) = \{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}$ and*

$$E(Bar_{n,t}) = E(K_n) \cup E(K_n) \cup \{u_i v_j | 1 \leq i \leq n-1, 1 \leq j \leq n-1\},$$

where $\left| \{u_i v_j | 1 \leq i \leq n-1, 1 \leq j \leq n-1\} \right| = t$.

As examples see two non-isomorphic graphs $Bar_{3,2}$ in Figure 4. Notice that B_{n-1}^c is one of the specific case of $B_{n,(n-1)(n-2)}$. The left graph in Figure 4, is B_2^c .

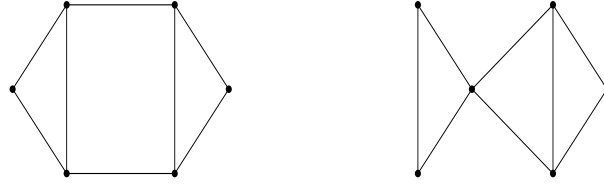


Figure 4: Two generalized barbell graphs $Bar_{3,2}$.

We have the following theorem.

Theorem 2.10 *For every $n \geq 3$ and $n \in \mathbb{N}$,*

$$D(Bar_{n,t}, x) = ((1+x)^n - 1)^2.$$

Proof. We prove this Theorem by induction on t . Suppose that $t = 1$, Then by Theorem 2.7, the result holds. Assume that the result holds for $t = (n-1)^2 - 1$. Let $t = (n-1)^2$ and e be the additional edge of $Bar_{n,t}$ to the $Bar_{n,t-1}$. By Theorem 2.5 two end vertices of edge e are domination-covered in $Bar_n - e$. So, by Theorem 2.6 the edge e is an irrelevant edge of $Bar_{n,t-1}$. Therefore by the induction hypothesis we have the result. \square

The following corollary is an immediate consequence of Theorems 2.7 and 2.10.

Corollary 2.11 *For each natural number n and $t \leq (n-1)^2$, Bar_n and $Bar_{n,t}$ have the same domination polynomial.*

The following example shows that, except for the generalized barbell graphs, there are other graphs in \mathcal{D} -equivalence classes of Bar_n .

Example 2.12 *All connected graphs in $[Bar_3]$ are the graphs Bar_3 , $Bar_{3,2}$, $Bar_{3,3}$, $Bar_{3,4}$ and two graphs in Figure 5.*

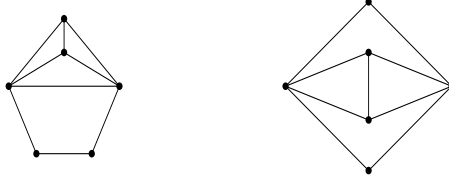


Figure 5: Two graphs in $[Bar_3]$.

3 Some graphs in $[K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_k}]$

We observed that, for each natural number n and $t \leq (n-1)^2$, the domination polynomials of Bar_n and $Bar_{n,t}$ is $((1+x)^n - 1)^2$. In this section, we present graphs whose domination polynomials are $\prod_{i=1}^k ((1+x)^{n_i} - 1)$. For this purpose, we construct families of graphs from a path P_k which we denote them by $S(G_1, G_2, \dots, G_k)$ in the following definition.

Definition 3.1 *The graph $S(G_1, G_2, \dots, G_k)$ is a graph which obtain from a path P_k with the vertices $\{v_1, v_2, \dots, v_k\}$, by substituting a graph G_i of order $n_i \geq 3$, for every vertex v_i of P_k , such that*

- *for $i = 1, k$, the graphs G_i have at least one vertex of degree $n_i - 1$ and other G_i 's have at least two vertices of degree $n_i - 1$, and*
- *in the graph $S(G_1, G_2, \dots, G_k)$, the end vertices of each edge e_i in the path graph, P_k are one vertex of degree $n_i - 1$ in graphs G_{i-1} and G_i .*

We have the following result for graph $S(G_1, G_2, \dots, G_k)$.

Theorem 3.2 *For every natural number $k \geq 2$,*

$$D(S(G_1, G_2, \dots, G_k), x) = D(G_1, x)D(G_2, x) \dots D(G_k, x).$$

In particular if $G_i = K_{n_i}$ and $n_i \geq 3$, then

$$D(S(K_{n_1}, K_{n_2}, \dots, K_{n_k}), x) = \prod_{i=1}^k D(K_{n_i}, x) = \prod_{i=1}^k ((1+x)^{n_i} - 1).$$

Proof. Let e_i ($1 \leq i \leq k$) be the edge joining G_{i-1} and G_i in $S(G_1, G_2, \dots, G_k)$. By Theorem 2.5 two end vertices of edge e_i are domination-covered in $S(G_1, G_2, \dots, G_k) - e_i$. So, by Theorem 2.6 every edge e_i is an irrelevant edge of $S(G_1, G_2, \dots, G_k)$. Therefore we have the result. \square

We shall generalize the graphs $S(G_1, G_2, \dots, G_k)$ in Definition 3.1 such that this generalized graphs and $S(G_1, G_2, \dots, G_k)$ have the same domination polynomial. Suppose that $GS_t(K_{n_1}, K_{n_2}, \dots, K_{n_k})$ be a family of graphs in the form of $S(K_{n_1}, K_{n_2}, \dots, K_{n_k})$ such that the complete graphs K_{n_i} with $V(K_{n_i}) = \{u_1, \dots, u_{n_i}\}$ and $K_{n_{i+1}}$ with $V(K_{n_{i+1}}) = \{v_1, \dots, v_{n_{i+1}}\}$ are joined with t following edges

$$\{u_i v_j | 1 \leq i \leq n_i - 1, 1 \leq j \leq n_{i+1} - 1\}.$$

Similar to the proof of the Theorem 2.10, we have the following theorem:

Theorem 3.3 *For each natural number t , all graphs in the family of $GS_t(K_{n_1}, K_{n_2}, \dots, K_{n_k})$ have the same domination polynomial. More precisely, the domination polynomial of each H in $GS_t(K_{n_1}, K_{n_2}, \dots, K_{n_k})$ is equal to $\prod_{i=1}^k ((1+x)^{n_i} - 1)$.*

Conclusion. In this paper, we studied the \mathcal{D} -equivalence classes of barbell graphs Bar_n . We showed that, for each natural number n , $2K_n$, Bar_n , $Bar_{n,t}$ and the complement of the book graph of order $n-1$, B_{n-1}^c have the same domination polynomial, i.e., $[Bar_n] = [Bar_{n,t}] = [B_{n-1}^c] = [K_n \cup K_n]$. Example 2.12, implies that except for these kind of graphs, there are another graphs in this class. Therefore, exact characterization of graphs in $[Bar_n]$ remains as an open problem. Also we presented many families of graphs whose are in $[K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_k}]$, but similar to Example 2.12, there are another graphs in this class. So, exact characterization of graphs in $[K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_k}]$ remains as another open problem.

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